Announcements

1) Exam tomorrow
   - covers up to 2/18

2) Office hours cancelled for tomorrow because:

Colloquium, 4-5

CB 2062

Restatement:

$\exists\ \varepsilon$ with

$|\varepsilon| < \varepsilon_{machine}$,

$$fl(x) = (1+\varepsilon)x$$

(write out: $fl(x) = x'$

$$x' = x + \varepsilon x$$

$$x' - x = \varepsilon x \ )$$

Floating Point Arithmetic

We have operations

$\oplus, \ominus, \otimes, \oslash$

defined from

$$\mathbb{R} \times \mathbb{R} \to \mathbb{F}$$

(ideally)

except for $\oslash$, which maps from $\mathbb{R} \times (\mathbb{R} - \{0\}) \to \mathbb{F}$.

With the defining property:

$\exists \; \varepsilon$ with $|\varepsilon| < \varepsilon_{machine}$

$$\boxed{\begin{aligned} &1)\quad x \oplus y = (x+y)(1+\varepsilon) \\ &2)\quad x \otimes y = (x \times y)(1+\varepsilon) \\ &3)\quad x \ominus y = (x-y)(1+\varepsilon) \\ &4)\quad x \,\dot{\div}\, y = \left(\frac{x}{y}\right)(1+\varepsilon) \end{aligned}}$$

Example 1: (Choose an F)

$$F = \left\{ \frac{m}{16} \mid m \in \mathbb{Z} \right\}$$

$$fl(\pi) = \frac{50}{16}$$

$$\pi - \frac{50}{16} \approx .0166$$

$$\pi - \frac{51}{16} \approx -.0459$$

$$\pi \oplus e = \frac{94}{16}$$

since

$$\pi + e - 93/16 = .0474$$

and

$$\pi + e - 94/16 = -.0151$$

(Re) Definition: ($\varepsilon_{machine}$)

Define $\varepsilon_{machine}$ to be the smallest positive number ε satisfying:

1) $fl(x) = x(1+\varepsilon)$

2) $x \oplus y = (x+y)(1+\varepsilon)$

3) $x \ominus y = (x-y)(1+\varepsilon)$

4) $x \otimes y = (x \times y)(1+\varepsilon)$

5) $x \oslash y = \left(\frac{x}{y}\right)(1+\varepsilon)$

for all $x, y \in \mathbb{R}$

$(y \neq 0$ in 5$)$$)$.

Stability

Recall: A problem is simply a function $f: X \to Y$ where X and Y are normed (finite dimensional) vector spaces.

Definition: (algorithm)

An **algorithm** is simply another function

$\tilde{f} : X \to Y$ - but the codomain of $\tilde{f}$ is usually smaller than Y!

In our case,

Y is usually F

or "vector spaces

over F"

Definition (relative error)

The **relative error** of a given algorithm $\tilde{f} : X \to Y$ for a function $f : X \to Y$ is

$$\boxed{\frac{\| f(x) - \tilde{f}(x) \|}{\| x \|}}$$

Notation (Big O)

Given functions

$f, g : \mathbb{R} \to \mathbb{R}$,

we write

$$f(x) = O(g(x))$$

if $\exists \; c > 0$ such that

either $f(x) \leq c|g(x)|$

for $x \to 0$ or

$f(x) \leq c|g(x)|$ for $x \to \infty$.

We'll use $x \to 0$ at the beginning since we'll think of $x = \varepsilon_{machine}$ and we want small values of $\varepsilon_{machine}$.

Example 2: $\left(\frac{\sin x}{x}, \frac{\text{poly}}{e^x}\right)$

$\sin(x) = O(x)$ as

$x \to 0^+$ since

$\sin(x) \le x$.

(Here, $C = 1$)

If $p(x)$ is any polynomial, then

$$p(x) = O(e^x) \quad (x \to \infty)$$

since

$$\lim_{x \to \infty} \frac{p(x)}{e^x} = 0$$

(l'Hopital's rule)

Again $C = 1$.

Definition: (accuracy)

An algorithm $\tilde{f}: X \to Y$ for $f: X \to Y$ is said to be accurate if

$$\boxed{\frac{\|f(x)-\tilde{f}(x)\|}{\|x\|} < O(\varepsilon_{machine})}$$

General accuracy is usually too much to hope for, so we replace it with successively weaker properties.

Definition: (backwards stability)

An algorithm $\tilde{f}: X \to Y$ for $f: X \to Y$ is called **backwards stable** if

$\forall \; x \in X, \; \exists \; \tilde{x} \in X$

with $\boxed{f(\tilde{x}) = \tilde{f}(x)}$

for $\boxed{\dfrac{\|x - \tilde{x}\|}{\|x\|} = O(\varepsilon_{machine})}$

<u>Book:</u> "A backwards stable algorithm gives exactly the right answer to nearly the right question"

— you had to replace x by $\tilde{x}$

We like backwards stable algorithms!

Example 3 Every one of the floating point operations $\oplus, \ominus, \otimes, \oslash$, is backwards stable.